DYNAMIC LIMIT PRICING FOR THE MULTIPRODUCT FIRM

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The correction of Textual Errors (Courteous Reader) is a work of time, and that hath taken wing. The more faults thou findest, the larger field is presented to thy humanity to practise in. Be indulgent in thy censure, and remember that Error, whether Manual or Mental, is an inheritance, descending upon us, from the first of our Race.

-Francis Bacon, 1640
Introduction

The problem of the dominant firm and the possibility of entry deterrence was first examined by Joe S. Bain (1956) and Franco Modigliani (1958). Their work introduced the idea that a dominant firm would set a price for its product with consideration of the effect that such a price would have on entry into the market by a 'fringe' group of small, competitive firms. Specifically, the dominant firm enjoying a cost advantage over its rivals would price at the limit price, which was defined as that price at which entry was non-existent.

Among the first major developments of their ideas was the dynamic limit pricing model developed by Darius Gaskins (1971). This model used an optimal-control framework to show that a profit maximizing firm would balance between current profits, which would be maximized by charging a high price, and future market share, which would be maximized by charging a relatively low price—the limit price. In short, the firm acts to maximize the present value of current and future profits.

Gaskins' work deals only with one homogenous product. Obviously, such an analysis can not be meaningfully applied to many modern dominant firms that produce more than one good. This paper will develop a dynamic limit pricing model for a multiproduct firm facing interdependent demands for its products. Thus the firm now 'balances' profits not only across time, but also across goods. We will retain Gaskins' assumption that entry into a market is positively correlated with profits being earned by
those who are currently producing in the market, which is borne out by empirical work.\textsuperscript{2} We further assume that entry is atomistic—that is, that entry is made into markets independent of entry into others by the same firm or any others. This assumption implies single product entrant firms or entrant firms that (unlike the dominant firm) make decisions regarding their several products independently.
The Basic, Two-Good Model

For the sake of clarity in examining cross effects, we will first develop a model for a dominant firm that produces only two goods, \( x \) and \( y \). This analysis will later be expanded to the \( n \)-good case.

The demands for \( x \) and \( y \) are described by

\[
\begin{align*}
(1a) \quad q_x^i(p_x(t), p_y(t), t) &= f_x(p_x(t), p_y(t), t) - a_x(t) \\
(1b) \quad q_y^i(p_y(t), p_x(t), t) &= f_y(p_y(t), p_x(t), t) - a_y(t)
\end{align*}
\]

where \( a_i(t) \) equals the level of the fringe or entrants' sales of \( i \) at time \( t \). Only entry in the relevant good affects the demand curve faced by the dominant firm: there are no 'brand-specific' complement or substitute relationships.

Given these functions, the problem faced by the dominant firm is to choose \( p_x \) and \( p_y \) to maximize

\[
(2) V = \int_0^\infty \left[ (p_x(t) - c_x^i) q_x^i(p_x^i, p_y^i, t) + (p_y(t) - c_y^i) q_y^i(p_x^i, p_y^i, t) \right] e^{-rt} dt
\]

which is the present discounted value of current and future firm profits.

The rate of entry by fringe producers is \( e_i(t) \), and is determined by the expected rate of return. We will assume that entrants assume that current product price equals future product price; this is implied by the competitive nature of fringe firms. This assumption implies an entry function that is linear and monotonically non-decreasing:
Note that \( \overline{p}^x \), and similarly for \( y \). The limit price is \( \overline{p}^x \); i.e., \( p \geq \overline{e}(t) = 0 \). Thus, the dominant firm is assumed to not be at a cost disadvantage. The parameters \( k \) and \( l \) govern the rate of entrant 'adjustment' to the profits obtainable in the market.

Optimal control techniques can now be used to examine the optimal pricing strategy over time of the dominant firm. Equation (2) can be rewritten as

\[
(4) \int_0^\infty \left[ (p^x - c^x) (f^x(p^x, \rho^y, t) - a^x) + (p^y - c^y) (f^y(p^x, \rho^y) - a^y) \right] e^{-\beta t} dt
\]

subject to the conditions expressed in (3a) and (3b). In optimal control terms, \( a^x \) and \( a^y \) are the state variables, while \( p^x \) and \( p^y \) are the control variables.

The necessary conditions for optimization of \( V \) can, by use of Pontryagin's maximum principle, be stated in terms of the Hamiltonian:

\[
(5) H = \left[ (p^x - c^x) f^x(p^x, \rho^y, t) - a^x \right] + \left[ (p^y - c^y) f^y(p^x, \rho^y) - a^y \right] e^{-\beta t} + \lambda_1(t) (p^x - \overline{p}^x) + \lambda_2(t) (\rho^y - \overline{\rho}^y)
\]

The adjoint variables \( \lambda_1 \) and \( \lambda_2 \) equal \( \frac{\partial V}{\partial a^x(t)} \) and \( \frac{\partial V}{\partial a^y(t)} \) and have the economic meaning of the shadow prices of an additional unit of entry into the specified market at any point in time. Maximizing \( H \) intuitively involves balancing present and discounted future profits across both markets. The maximum principle yields the following necessary conditions on variables along the optimal path: \( (p^x, p^y, a^x, a^y, \lambda_1, \lambda_2) \).
The state equations:

\[(6a) \frac{d\lambda^x}{dt} = \frac{\partial^2 H}{\partial x^2} = k(p^x - \bar{p}^x)\]

\[(6b) \frac{d\lambda^y}{dt} = \frac{\partial^2 H}{\partial y^2} = \delta(p^y - \bar{p}^y)\]

The multiplier equations:

\[(7a) \frac{d\lambda_1}{dt} = \frac{\partial H}{\partial x} = -(-e^{-rt}(p^x - \bar{c}^x) + \lambda_1(0) + \lambda_2(0)) = e^{-rt}(p^x - \bar{c}^x)\]

\[(7b) \frac{d\lambda_2}{dt} = \frac{\partial H}{\partial y} = (-e^{-rt}(p^y - \bar{c}^y) + \lambda_1(0) + \lambda_2(0)) = e^{-rt}(p^y - \bar{c}^y)\]

and the optimality conditions:

\[(8a) \frac{\partial H}{\partial p^x} = (f^x(p^x, p^y) + p^x \frac{\partial f^x}{\partial p^x} - \bar{a}^x - c^x \frac{\partial f^x}{\partial p^y} + (p^y - \bar{c}^y) \frac{\partial f^y}{\partial p^x}) e^{-rt} + \lambda_1 e^0\]

\[(8b) \frac{\partial H}{\partial p^y} = (f^y(p^x, p^y) + f^y(p^x, p^y) + (p^y - \bar{c}^y) \frac{\partial f^y}{\partial p^y}\]

These last conditions hold as long as profit is a smooth concave function of price along each of the initial demand curves. This assumption implies the concavity of \(H\) with respect to \(p^i\) and \(a^i\), for all \(i\), and is sufficient for the existence of an optimal path.

The profit function can be written as:

\[(9) \Pi = (p^x - \bar{c}^x)f^x + (p^y - \bar{c}^y)f^y\]

It will be concave if and only if the following two conditions are met:

\[(10) \frac{\partial^2 \Pi}{\partial p^i^2} < 0, \text{ for } i = 1, 2\]

\[(11) \frac{\partial^2 \Pi}{\partial p^x^2} \cdot \frac{\partial^2 \Pi}{\partial p^y^2} - \left( \frac{\partial^2 \Pi}{\partial x \partial y} \right)^2 \geq 0\]
These conditions in turn imply:
\[ (12) \ (\rho^x - c^x) \frac{\partial^2 f^x}{\partial \rho^x^2} + 2 \frac{\partial f^x}{\partial \rho^x} \ (\rho^y - c^y) \frac{\partial^2 f^y}{\partial \rho^y^2} \leq 0 \]
since this condition must be met to ensure concavity.

The three sets of conditions derived from the Hamiltonian yield the simultaneous differential equations:

(13a) \[ \frac{\partial \lambda_1}{\partial t} = \mathcal{A} \left( \rho^x - c^x \right), \ \lambda_1(0) = \alpha_x^* \]

(13b) \[ \frac{\partial \lambda_2}{\partial t} = \mathcal{A} \left( \rho^y - c^y \right), \ \lambda_2(0) = \alpha_y^* \]

and

(14a) \[ \frac{\partial \lambda_1}{\partial t} = e^{-r^+} (\rho^x - c^x) \]

(14b) \[ \frac{\partial \lambda_2}{\partial t} = e^{-r^+} (\rho^y - c^y) \]

where

(15a) \[ \lambda_1 = (\alpha_x + \epsilon \sigma (\rho^x - c^x) \frac{\partial f^y}{\partial \rho^x} - f^x - (\rho^y - c^y) \frac{\partial f^y}{\partial \rho^x} ) e^{-r^+/k} \]

(15b) \[ \lambda_2 = (\alpha_y + (\rho^y - c^y) \frac{\partial f^y}{\partial \rho^y} - (\rho^x - c^x) \frac{\partial f^y}{\partial \rho^y} - f^y ) e^{-r^+/s} \]

We can eliminate \( \lambda_1 \) and \( \lambda_2 \) from these equations and write the necessary conditions as simultaneous equations in \( p^1(t) \) and \( a^1(t) \):

(13a) and (13b) and

(16a) \[ \frac{\partial p^x}{\partial t} = \mathcal{A} \left( \rho^x - c^x \right) + r \left( \sigma^x - (\rho^x - c^x) \frac{\partial f^x}{\partial \rho^x} - f^x - (\rho^y - c^y) \frac{\partial f^y}{\partial \rho^x} \right) \frac{\partial^2 f^x}{\partial \rho^x^2} - 2 \frac{\partial f^x}{\partial \rho^x} - (\rho^x - c^x) \frac{\partial^2 f^x}{\partial \rho^x^2} - (\rho^y - c^y) \frac{\partial^2 f^y}{\partial \rho^y^2} \]

(16b) \[ \frac{\partial p^y}{\partial t} = \mathcal{A} \left( \rho^y - c^y \right) + r \left( \sigma^y - (\rho^y - c^y) \frac{\partial f^y}{\partial \rho^y} - f^y - (\rho^x - c^x) \frac{\partial f^x}{\partial \rho^y} \right) \frac{\partial^2 f^y}{\partial \rho^y^2} - 2 \frac{\partial f^y}{\partial \rho^y} - (\rho^y - c^y) \frac{\partial^2 f^y}{\partial \rho^y^2} \]

and the terminal conditions on \( a^x^* \) and \( a^y^* \), which cannot be incl-
uded in the simultaneous equations:

(17a) \( \lim_{t \to 0^+} a_y(t) = 0 \)

(17b) \( \lim_{t \to 0^+} a_y(t) = 0 \)
Comparative Statics and the Optimal Trajectory

We will now determine the effects of shifts in the model parameters on the optimal path. We need to examine the equilibrium level of entrant output, $a^x$ and $a^y$, and will now develop expressions for these equilibrium values. We can set equations (13a), (13b), (16a), (16b) equal to zero and solve simultaneously which yields

$$
\begin{align*}
(18a) \quad & a^x = (p^x - c^x) \frac{\partial f^y}{\partial p^x} + (p^y - c^y) \frac{\partial f^y}{\partial p^x} + f^y - \frac{k}{r} (p^x - c^x) \\
(18b) \quad & a^y = (p^y - c^y) \frac{\partial f^y}{\partial p^y} + (p^x - c^x) \frac{\partial f^x}{\partial p^y} + f^y - \frac{k}{r} (p^y - c^y)
\end{align*}
$$

We can also develop an expression for equilibrium market share. The dominant firm's share of the market at any point in time is $s(t) = \frac{f(p) - a}{f(p)}$. $s(t)$ will approach an equilibrium $s$ as $f$ and $a$ approach their respective equilibria. Thus, we can think of the firm's optimal strategy as pricing to maximize the long-run optimal market shares:

$$
\begin{align*}
(19a) \quad & \hat{s}^x = \left( \frac{f^x}{f^x} - \frac{f^y}{f^y} \right) \frac{\partial f^x}{\partial p^x} - \frac{f^y}{f^y} \frac{\partial f^x}{\partial p^x} + \frac{f^x}{f^x} \frac{\partial f^y}{\partial p^y} \\
(19b) \quad & \hat{s}^y = \left( \frac{f^x}{f^x} - \frac{f^y}{f^y} \right) \frac{\partial f^y}{\partial p^y} - \frac{f^x}{f^x} \frac{\partial f^y}{\partial p^y}
\end{align*}
$$

The signs of the comparative statics derivatives depend upon the assumed relationship between the two goods. We will assume here that they are substitutes; the case of complementary goods will be discussed later. The derivatives are:

$$
(a) \quad \frac{\partial \hat{s}^x}{\partial p^x} = (p^x - c^x) \frac{\partial^2 f^x}{\partial p^x^2} + (p^y - c^y) \frac{\partial^2 f^y}{\partial p^x^2} + 2 \frac{\partial^2 f^y}{\partial p^x^2} - \frac{k}{r} \quad \triangleq 0
$$
(b) \( \frac{\partial^2 x}{\partial p \partial y} = (\rho^x - c^x) \frac{\partial^2 y}{\partial p^2} + (\rho^x - c^x) \frac{\partial^2 y}{\partial p \partial y} + \frac{\partial f^x}{\partial p} + \frac{\partial f^y}{\partial p} \)

(c) \( \frac{\partial^2 x}{\partial c \partial x} = \frac{-2f^x}{p} + \frac{k}{r} > 0 \Rightarrow \frac{\partial^2 x}{\partial c^x} < 0 \)

(d) \( \frac{\partial^2 x}{\partial c^y} = \frac{-2f^y}{p} \sqrt{< 0} \Rightarrow \frac{\partial^2 x}{\partial c^y} > 0 \)

(e) \( \frac{\partial^2 x}{\partial r \partial c^x} = \frac{k}{r^2} (\rho^x - c^x) > 0 \Rightarrow \frac{\partial^2 x}{\partial r \partial c^x} \leq 0 \)

(f) \( \text{sgn} \left[ \frac{\partial^2 x}{\partial p^2} \right] = \text{sgn} \left[ f^x \left( \frac{k}{r} - (\rho^x - c^x) \frac{\partial f^x}{\partial p^2} - \frac{\partial f^x}{\partial p} - (\rho^y - c^y) \frac{\partial f^y}{\partial p^2} \right) \right] \right) - \frac{\partial f^x}{\partial p} \left( \frac{k}{r} (\rho^x - c^x) - (\rho^y - c^y) \frac{\partial f^x}{\partial p} - (\rho^x - c^x) \frac{\partial f^y}{\partial p} \right) \right] \)

(g) \( \text{sgn} \left[ \frac{\partial^2 x}{\partial p \partial y} \right] = \text{sgn} \left[ f^x \left( (\rho^x - c^x) \frac{\partial f^x}{\partial p \partial y} - (\rho^y - c^y) \frac{\partial f^y}{\partial p \partial y} - \frac{\partial f^y}{\partial p} \right) \right] \right) - \frac{\partial f^x}{\partial p} \left( \frac{k}{r} (\rho^x - c^x) - (\rho^y - c^y) \frac{\partial f^x}{\partial p} - (\rho^x - c^x) \frac{\partial f^y}{\partial p} \right) \right] \)

(h) \( \frac{\partial^2 x}{\partial k \partial c^x} = \frac{k}{r} (\rho^x - c^x) \leq 0 \Rightarrow \frac{\partial^2 x}{\partial k \partial c^x} > 0 \)

(i) \( \frac{\partial^2 x}{\partial a^x} = \frac{\partial^2 x}{\partial a^x} = \frac{\partial^2 x}{\partial a^y} = \frac{\partial^2 x}{\partial a^y} = \frac{\partial^2 x}{\partial a^y} = \frac{\partial^2 x}{\partial a^y} = 0 \)

And now, for \( y \), the derivatives are:

(j) \( \frac{\partial^2 y}{\partial p^2} = (\rho^y - c^y) \frac{\partial^2 y}{\partial p^2} + 2 \frac{\partial f^y}{\partial p} + (\rho^x - c^x) \frac{\partial^2 y}{\partial p^2} - \frac{\partial f^y}{\partial p} \)

(k) \( \frac{\partial^2 y}{\partial p^x} = (\rho^y - c^y) \frac{\partial^2 y}{\partial p^2} + (\rho^x - c^x) \frac{\partial^2 y}{\partial p^2} + \frac{\partial f^y}{\partial p^2} + \frac{\partial f^y}{\partial p^x} \)

(l) \( \frac{\partial^2 y}{\partial c^y} = \frac{-\partial f^y}{c^y} + \frac{j}{r} \geq 0 \Rightarrow \frac{\partial^2 y}{\partial c^y} < 0 \)

(m) \( \frac{\partial^2 y}{\partial c^x} = \frac{-\partial f^x}{c^x} < 0 \Rightarrow \frac{\partial^2 y}{\partial c^x} \geq 0 \)

(n) \( \frac{\partial^2 y}{\partial r^2} = \frac{k}{r^2} (\rho^y - c^y) > 0 \Rightarrow \frac{\partial^2 y}{\partial r^2} \leq 0 \)
Derivative (a), thanks to the required convexity of the profit function, is negative. Equations (a) and (c) together lead to the conclusion that as a dominant firm's cost advantage over the 'fringe' increases, its optimum level of fringe entry into the market decreases. Under the conditions specified for (a), (f) is likely to be positive. Here, to ensure this result, 

\[-(p^x - c^x) \frac{\partial^2 f^x}{\partial p^x \partial p^y},\]

as the only negative term, must be relatively small. Then (f) and (c) together imply that as the cost advantage increases, the dominant firm’s optimal market share increases as well.

The signs of equations (b) and (g) depend upon the signs of the cross terms \(\frac{\partial^2 f^x}{\partial p^x \partial p^y}\) and \(\frac{\partial^2 f^y}{\partial p^x \partial p^y}\). Given the assumption that the two goods are substitutes, i.e., that \(\frac{\partial f^x}{\partial p^y} < 0\) and \(\frac{\partial f^y}{\partial p^y} > 0\), the best assumptions about the second derivatives, \(\frac{\partial^2 f^x}{\partial p^x \partial p^y} > 0\), \(\frac{\partial^2 f^y}{\partial p^x \partial p^y} > 0\) lead to the conclusion that the signs of the second cross-partial that we are concerned with are indefinite. The only condition on their values is that given by the convexity conditions examined before, and it does not give us the signs.

In (b), if the two partials are either positive, or is the terms
containing them are small, we reach the conclusion that (b) is positive. This together with (d) indicates that an increase in cost advantage in one market will produce an increase in the dominant firm's optimal level of fringe entry in the other market. Equation (g) similarly shows that negative signs for the two cross partials, and/or a small value for the $(\frac{\partial^2}{\partial \rho^x \partial \rho^y})$ and $(\frac{\partial^2}{\partial \rho^y \partial \rho^x})$ terms, indicate that (g) will be positive. Taken together with (d) this shows that an increase in cost advantage in one market leads to a decrease in optimal market share in the second market. One conclusion to be drawn from this is that the validity of the intuitive results about the effect of cost advantages on the firm's choice of market share depends upon the existence of demand surfaces that are relatively flat. High curvature results in large second partials, which, if they are positive, will reverse many of the above results.

Equation (e) yields the unsurprising result that an increase in the discount rate will increase optimal entry and decrease optimal market share. An increase in the discount rate reflects a relative preference shift towards short run profits, so the firm will choose a higher level of such profits as against the higher long run profits that could be earned by keeping price low and the market share high. Equation (h) shows that as the responsiveness coefficient $k$ increases, entry levels go down and market share increases. As Gaskins points out, this counter-intuitive result has some interesting policy implications. Policies that attempt to reduce the extent of oligopoly by lowering barriers to entry (and exit) may lead the dominant firm to price to drive out entry. Equation (i) simply says that the start-
ing level of entry does not affect the optimal solution to the problem facing the dominant firm.

The market for $y$ is discussed in the second set of equations. The results parallel those discussed for $x$. It is interesting to note that (b) is identical to (k). Evidently the cross-effect of limit price on the other market is independent of the direction of the effect. This result holds even though the demands, response parameters, etc. need in no way be equal. The same is not true for market share, interestingly.

What happens if $x$ and $y$ are complements? Some derivatives will not change in sign: they include $\frac{\partial f^x}{\partial p^x}$, $\frac{\partial^2 f^x}{\partial p^x \partial p^x}$, $\frac{\partial f^y}{\partial p^y}$, $\frac{\partial^2 f^y}{\partial p^y \partial p^y}$.

Some will switch in sign: $\frac{\partial f^y}{\partial p^x}$, $\frac{\partial^2 f^y}{\partial p^x \partial p^x}$, $\frac{\partial^2 f^y}{\partial p^y \partial p^y}$. Importantly, the two cross second partial terms, $\frac{\partial^2 f^x}{\partial p^x \partial p^y}$ and $\frac{\partial^2 f^y}{\partial p^y \partial p^x}$, are now going to be positive. All of this will, of course, change the signs of some of the comparative statics derivatives. The following equations will remain unchanged in sign: (a), (c), (h), and (i). As before, parallel results obtain for $y$. In (f), a stricter set of assumptions on the curvature of demand and/or the cost advantages is needed to preserve the earlier result that $\frac{\partial s^x}{\partial p^x}$ is positive. Since (c) remains unchanged, it is now more likely that the counter-intuitive result—an increase in cost advantage leading to a decrease in relevant market share—will be found, thanks to the effects of price change on the other market and hence on firm profits.

Equation (b) is again indeterminate, with the sign depending on the magnitudes of cost advantage and demand curvature versus the slopes of the demands. The intuitive result that (b) is negative will hold if the first elements are small relative to the second. Equation (g) will be positive under similar but stricter
conditions. Together with (d), it shows that as cost advantage in one market increases, allowed entry in the other market will decrease, and equilibrium market share will increase (a reversal of the earlier results) as long as the demands are relatively flat. As before, the market for y parallels the market for x; and the identity of (b) and (k) still holds.

We have shown that the analysis of changes in limit prices and costs is more complex than is intuitively apparent: whether the two goods are complements or substitutes, the presence of cross effects between the two markets through the demand functions leads to the conclusion that small second derivatives—'flat' demand surfaces—are needed to support the intuitive results, particularly if the dominant firm enjoys a large cost advantage.
Extending The Model To N Goods

We will now extend the model we have developed for two goods to the case of the firm producing a set of goods $N$, with $n$ elements. $N=(x^1, x^2, \ldots, x^n)$. The demand curve for good $i$ will then be of the form

\[ (1) \quad q^i(t)(p^i(t), p^2(t), \ldots, p^n(t), t) = f^i(p^i(t), \ldots, p^n(t), t) \]

where the variables have the same meaning as in the previous sections. Given this set of demand functions faced by the dominant firm, the firm will choose a vector of prices that will maximize

\[ (2) \quad V = \int_0^\infty \sum_{i=1}^n \left( (\rho^i(t)-\bar{c}^i) q^i(\bar{p}, t) e^{-rt} \right) dt \]

As before, the firm will act to maximize the present value of current and future profits.

The rate of entry by fringe producers, \( \frac{\partial a^i(t)}{\partial t} = q^i(t) \) is again proportional to the difference between the current and limit prices. Again, entrants make the assumption that future product price will equal current product price. Hence,

\[ (3) \quad \frac{\partial a^i}{\partial t} = q^i(t) = k^i(p^i(t)-\bar{p}) \quad \bar{a}^i(0) = a^i_0 \quad \bar{p}^i \geq \bar{c}^i \]

And the dominant firm is assumed to not be at a cost disadvantage with respect to the fringe entrants.

Once again we will use optimal control techniques to examine the best pricing strategy for the dominant firm. The firm will maximize

\[ (4) \quad V = \int_0^\infty \sum_{i=1}^n \left[ (\rho^i(t)-\bar{c}^i) \left( f^i(\bar{p})-a^i(t) \right) \right] e^{-rt} dt \]
subject to (3), where the $a^i$'s are the state variables and the $p^i$'s are the control variables.

The Hamiltonian then becomes

$$H = \sum c_i \left[ (\rho^i - c^i) \left( f^i - a^i \right) \right] e^{-rt} + \lambda_i k^i (\rho^i - p^i)$$

which generates the following necessary conditions to be met by the maximizing values $(\rho^i, a^i, \ldots, a^n)$ and $(\lambda^i, \ldots, \lambda^n)$

The state equations:

$$\frac{d a^i}{dt} = \frac{\partial H}{\partial \lambda_i} = k^i (\rho^i - p^i)$$

The multiplier equations:

$$\frac{d \lambda_i}{dt} = - \frac{\partial H}{\partial a^i} = e^{-rt} (\rho^i - c^i)$$

And the optimality conditions:

$$\frac{\partial H}{\partial \rho^i} = \left( f^i + (\rho^i \frac{\partial f^i}{\partial \rho^i} - c^i) \frac{\partial f^i}{\partial \rho^i} - a^i - \sum (\rho^j - c^j) \frac{\partial f^j}{\partial \rho^i} \right) e^{-rt}$$

$$+ \lambda_i k^i = 0$$

The boundary conditions are $a^i(t_0)$ given, $i(t_1) = 0$. And $p^i(t)$ must maximize $H(t, a^i(t), \rho^i, \lambda^i(t))$ with respect to $u_i$ at each $t$.

These necessary conditions yield the following differential equations upon solution:

$$\frac{d a^i}{dt} = k^i (\rho^i - p^i) \quad a^i(0) = a^i_0$$

$$\frac{d \lambda^i}{dt} = e^{-rt} (\rho^i - c^i)$$

where

$$\lambda_i = (a^i + c^i \frac{\partial f^i}{\partial \rho^i} - f^i - \rho^i \frac{\partial f^i}{\partial \rho^i} - \sum (\rho^j - c^j) \frac{\partial f^j}{\partial \rho^i}) e^{-rt} / k^i$$
\( A \) can be eliminated from these equations and the necessary conditions written as simultaneous differential equations in \( p^i(t) \) and \( a^i(t) \):

\[
(12) \frac{\partial a^i}{\partial t} = k^i(p^i - c^i), \quad a^i(0) = a_0^i
\]

\[
(13) \frac{\partial p^i}{\partial t} = k^i(p^i - c^i) + r(a^i - (p^i - c^i)) \frac{\partial f^i}{\partial p^i} - \sum_{j=1}^{n} (p^j - c^j) \frac{\partial f^j}{\partial p^i} - f^i
\]

These 2n equations then generate a family of trajectories in \( a^1, \ldots, a^n, p^1, \ldots, p^n \) space.

To guarantee the existence of an optimal path, profit must again be a smooth concave function of price along the original demand curves. This implies that the Hessian of second partials will be negative semi-definite, which in turn implies that and the principle minors will alternate in sign. To satisfy these conditions, the following must hold:

\[
(14) \frac{\partial^2 \Pi}{\partial p^i \partial p^j} = \sum_{i=1}^{n} (p^i - c^i) \frac{\partial^2 f^i}{\partial p^i \partial p^j} + 2 \frac{\partial f^i}{\partial p^j} \leq 0
\]
Once again we will use comparative statics to determine what a shift in a model parameter will do to the optimal path. We will need to examine the equilibrium level of entrant output $a^1_i$.

An expression for $a^1_i$ can be found by setting equations (12) and (13) equal to zero and solving simultaneously, which gives us

$$a^1_i = \left( \frac{b}{r} \left( \bar{p}^i - c^i \right) - \sum_j \left( \bar{p}^j - c^j \right) \frac{\partial f^i_j}{\partial \bar{p}^i_j} \right) - \frac{f^i}{r} \left( \bar{p}^i - c^i \right)$$

The expression for the long-run optimal market share $s^1_i$, which will be reached as $f^i$ and $a^i$ approach their respective equilibria $\mathbf{v}_i$ is

$$s^1_i = \left( \frac{b}{r} \left( \bar{p}^i - c^i \right) - \sum_j \left( \bar{p}^j - c^j \right) \frac{\partial f^i_j}{\partial \bar{p}^i_j} \right) - \frac{f^i}{r} \left( \bar{p}^i - c^i \right)$$

Formerly, we made the assumption that the two goods were either substitutes or complements. Such an assumption cannot be as easily made in the $n$-good case, as some of the goods will be complements and others substitutes. Therefore, expressions and not signs will be given for most of the following partial derivatives:

(a) $\frac{\partial \hat{s}^i}{\partial \bar{p}^i} = \left( \frac{b}{r} \left( \bar{p}^i - c^i \right) \frac{\partial^2 f^i}{\partial \bar{p}^i \partial \bar{p}^i} + \frac{\partial f^i}{\partial \bar{p}^i} \right) + \sum_j \left( \frac{\partial f^i_j}{\partial \bar{p}^i} \frac{\partial^2 f^j}{\partial \bar{p}^i \partial \bar{p}^j} \right) + \frac{\partial f^i}{\partial \bar{p}^i} - \frac{k^i}{r}$

(b) $\frac{\partial \hat{s}^i}{\partial \bar{p}^k} = \left( \frac{b}{r} \left( \bar{p}^i - c^i \right) \frac{\partial^2 f^i}{\partial \bar{p}^i \partial \bar{p}^k} \right) + \sum_j \left( \frac{\partial f^i_j}{\partial \bar{p}^i} \frac{\partial^2 f^j}{\partial \bar{p}^i \partial \bar{p}^k} \right) + \frac{\partial f^i}{\partial \bar{p}^k} + \frac{\partial f^k}{\partial \bar{p}^i}$

(c) $\frac{\partial \hat{s}^i}{\partial c^i} = -\frac{\partial f^i}{\partial c^i} + \frac{k^i}{r} > 0 \Rightarrow \frac{\partial \hat{s}^i}{\partial c^i} < 0$

(d) $\frac{\partial \hat{s}^i}{\partial c^k} = -\frac{\partial f^k}{\partial c^i}$

(e) $\frac{\partial \hat{s}^i}{\partial r} = \frac{k^i}{r^2} \left( \bar{p}^i - c^i \right) \geq 0 \Rightarrow \frac{\partial \hat{s}^i}{\partial r} \leq 0$
If we are dealing with \( n \) goods, there are three cases to be discussed when evaluating these comparative statics results. The first two cases are straightforward; all \( n \) goods could be substitutes for each other or they could all be complements. If either of these simple cases holds, the analysis would be basically identical with that presented in the two-good case. The conditions there used in connection with the comparative statics of the equilibrium market share would have to be extended to \( n \) goods, of course.

But what if some of the goods are substitutes to each other and others are complements? This third case can best be understood by thinking of some subset \( M = (x^1, \ldots, x^m) \) of the set of goods \( N \) as types of machines, cars, etc. These \( m \) goods are substitutes for each other: i.e., \( \frac{\partial^2 f^i}{\partial p^j \partial p^k} < 0 \) \( \forall i, j \in M \) and \( M \) is complementary to \( M \): \( \frac{\partial^2 f^i}{\partial p^i \partial p^j} > 0 \) \( \forall i \in C, j \in M \)

The members of the set \( C \) would include fuel, spare parts, and
other adjunct supplies and accessories. The relationship between members of \( C \) is more problematical. It is conceivable that they could be complements, substitutes, or have no realationship at all to each other. Since the concavity conditions place no constraints on the relationships between members of \( C \), the choice must be made, where possible, with reference to the industry involved. In the plain paper copier industry, the major members of \( C \) are paper, toner, replacement parts, etc.; they are probably complementary to each other in actual use: \[ \frac{\partial f^i}{\partial p^j} \leq 0, \quad \frac{\partial^2 f^i}{\partial p^j^2} \geq 0 \quad \forall \; i,j \in C. \]

With this discussion behind us, we can now embark upon the evaluation of the comparative statics derivatives in the \( n \)-good case.

As has already been discussed, the existence of an optimal path depends upon the concavity of the profit function with respect to price. The conditions necessary for concavity also are sufficient for \( (a) \) to be negative for all \( i \). These conditions are however not enough to show \( (f) \) positive. That result will be reached only if the demand surface is relatively flat (so that the second derivatives are small) or if the firm enjoys only a small across-the-board cost advantage (the \( (\bar{p}^i - c^i) \) vector is small in magnitude). Equation \( (c) \) is negative for all \( i \), so \( (a) \), \( (f) \), and \( (c) \) together show that for all goods, complements or substitutes, an increase in cost advantage leads to a decrease in entrant production and an increase in the dominant firm's market share in that market.

In evaluating \( (b) \) no such easy outs appear. While concavity does place restrictions on \( (b) \) \( \prod_i \prod_j \left( \frac{\partial^2 f}{\partial p^i \partial p^j} \right) \geq 0 \) this condition, once expanded and collected, is not helpful. If \( i \) and
k are elements of \( M \), then if the demand plane is relatively flat or cost advantages are small, \( (b) \) will be positive. If \( i \) and \( k \) are elements of \( C \), then the same conditions give us \( (b) \) as negative. If \( i \) is an element of \( M \) and \( k \) is an element of \( C \), or vice versa, \( (b) \) is again negative. Under similar, but stricter conditions on the cross partials and cost advantage vector, \( (g) \) is opposite \( (b) \) in sign for each of the above cases. Equation \( (d) \) will be negative when the two goods are both elements of \( M \); under the other two cases, \( (d) \) is negative.

All of this indicated that an increase in cost advantage for one good will increase optimal production by the fringe in the market for substitute goods, if any exist, while reducing the dominant firm's optimal market share in such markets. But it will decrease entrant production and increase market share in all other markets. Thus an increase in \( (p^i - c^i) \) for \( i \) an element of \( C \) will increase optimal market share in all markets, where if \( i \) is an element of \( M \), market share will increase in some goods and decrease in others.

Equations \((e), (h), \) and \((i)\) do not involve any cross effects between goods, and the analysis is identical to that presented in the two-good case.
Conclusion

This paper has attempted to extend an analysis similar to that presented by Gaskins to the more general n-good case. While there are only a few definite results stemming from this analysis, with a wider class of results, including some of the most interesting, resting upon assumptions that may or may not be warranted, the relatively simple form of analysis used in this paper has shown itself capable of shedding light on the behavior of the multi-product firm. In addition, several of the assumptions made in connection with the conclusions drawn in this paper may well be in line with the real world situation; specifically, the dominant firm probably does not enjoy a large cost advantage in many of its products. Minimum optimal scale is fairly small for a wide range of goods, and in this age of corporate espionage, production processes do not remain entirely secret for long. But issues of the applicability of a model such as this one can only be solved by applying it to a real-world problem; and that is another task.
This statement is somewhat simplistic. Depending on the price elasticity of demand and the entrants' cost structure, which will determine the short-run profit maximizing and limit prices respectively, it is possible that the two prices could be equal— or the limit price could be greater than the short-run price. Such cases will, however, be rare. As far as I am aware, no empirical examples have been found of such conditions.

Mansfield (1962) shows that the variation in rate of firms entering or exiting an industry is positively correlated with the level of industry profits.

If it were, its market share would continuously decline. If it is at a disadvantage in one market but not in another, conceivably it might stay in the 'loser' market—but under the conditions assumed in this paper, it is difficult to see why.

See Salop (1979) and Flaherty (1980) for discussions of barriers to entry and market signals by the dominant firm.
Bibliography


