A Model of Advertising:
The Relationship Between
Firms and Consumers

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INTRODUCTION AND BACKGROUND

Economic models serve to demonstrate how the market performs, given a set of conditions and assumptions. To be useful in policy and economic decision-making, the assumptions that are set forth should approximate reality. For some time, studies of the market assumed the conditions of price and quality homogeneity and perfect information by both consumers and firms. However, simple observation of the market shows that those assumptions do not approximate reality. We have all experienced different qualities of the same good between brands, are not certain of a product's quality before we buy it, and even after having bought and used the product several times we are not certain the product we are buying is the highest quality available. As a result of quality heterogeneity and consumer's imperfect assessment of quality, firms advertise in an effort to convince consumers that their product is the best buy they can make. This paper will examine the role of advertising in a market that incorporates the characteristics of quality dispersion, price homogeneity and consumers who learn imperfectly but always want to purchase the highest quality they can find. This study will show why firms choose to advertise, the indications of quality that advertising gives, and how the amount of advertising used by firms is a function of the quality of the product they produce.

The first economist to consider the implications of imperfect information in the market was Stigler in "The Economics of Information" [1961]. In this paper he introduced the topic and stated the basic premises
from which all future research and analysis have sprung. The basic elements of his model are price variation, quality homogeneity, and advertising by firms. The restriction of heterogeneity to a factor that can be assessed without uncertainty before purchase reduces the problem to a search problem. He included advertising as an element that can help reduce the cost of the consumer's search by identifying each seller's offer price. He makes the reasonable assumption that price advertising provides reliable information. The model he creates makes the advertising decision a function of how many consumers the advertisement will reach and the profit per consumer reached. Some of Stigler's other observations that should be considered in studies of the market and imperfect information are entry and exit by consumers, the learning and forgetting of information by consumers, and the obsolescence of information. He attributes price dispersion to these causes.

Nelson was responsible for the next major advancements in the market performance with imperfect information. One of these contributions was making the distinction between search goods, which are goods whose quality can be completely assessed prior to purchase, and experience goods, which are goods whose quality can only be determined by the consumer after it has been purchased and used [Nelson, 1970]. "Advertising as Information" [Nelson, 1974] examined the role of advertising in a market that includes imperfect information. One assertion he made which has generally been accepted is that there will be very little false advertising for search goods, because the advertisement's validity can be verified by the customer before purchasing it; however,
the content of advertisements for experience goods is of questionable validity, since the customer cannot establish the accuracy of the advertisement's statements until after purchasing and using the product. The conclusion that was reached by Nelson in this paper is that advertising provides the consumer with information that the firm is producing a good enough product that the repeat purchases of satisfied customers will make advertising a profitable activity for the firm. This initiated the study of advertising as a means of building the firm's reputation and the role of the firm's reputation in its profitability and quality production decisions.

This general background spawned two papers which have directly influenced the model presented in this paper. The first of these is "Reputation and Product Quality" [Rogerson, 1983]. The model developed by Rogerson is a general model of a market involving two qualities of the same product and a single market price. He also has provisions for consumers learning imperfectly about the quality of the product after having purchased it, capturing this notion in two mistake parameters, $\alpha$ and $\beta$. The first parameter is the probability that a purchaser of a high quality good will mistakenly identify it as a low quality good and the second parameter is the probability that a purchaser of a low quality good will mistakenly identify it as a high quality good. He specifies a departure rule and uses this to show that high quality firms will have more customers because they will have fewer dissatisfied customers who would leave the firm. The model also specifies profit functions such that the cost of producing a high quality good is more than the cost of producing a low quality good. The firms make a decision between
producing a high quality good, resulting in many customers with a low profit per customer, or producing a low quality good, resulting in few customers with a large profit per customer. His model considers advertising only as word of mouth advertising, which can be trusted and causes more customers to come to the high quality firms. The result of word of mouth advertising is that the reputation of the firm improves.

"Closed Form Equilibrium Price Distribution Functions" [Mortensen, 1985] presents a model of a market with price dispersion and quality homogeneity, advertising and learning by consumers. The consumers receive advertising messages based on the intensity with which each firm advertises. When they receive a message that shows where a lower priced good than the one they are currently purchasing is available, they will move to that good. Consumers randomly enter the market and purchase the lowest priced product they are aware of, and as they gather more information they move down the price distribution from high to low. Because the model includes a provision for births and deaths to the market, equilibrium can be found to exist when the flow into each price is the same as the flow out of that price. The inclusion of births and deaths in the model also enables the price dispersion to exist (otherwise, as consumers gather information they would leave the high priced firms and eventually purchase only from the lowest priced firms). Mortenson shows that there will exist a continuous price offer distribution from the competitive price to the monopoly price. The amount of advertising is dependent upon the price offered by the firm; the highest priced firms will have no incentive to advertise, while the lowest priced firms will
advertise, since every customer reached by their advertising who is not yet buying their product will begin to do so after being informed that the lower priced good exists.

CONSTRUCTION OF THE MODEL

The model presented in this paper borrows elements from both Rogerson's and Mortensen's models. It is placed in the context of a market for goods whose quality can only be determined with an element of uncertainty after purchase and use. The price received by each firm is the same, regardless of the quality of the product sold. For simplificity, this model will only consider a market with two firms, one producing a high quality product and the other producing a low quality product. Consumers want to purchase the higher quality good, but the only information they have is their own experience and the advertising by the firms. The firms are profit maximizing, but since there are only two firms, zero and equal profit conditions do not apply. The only condition is that the firms must have positive profits.

Consumer Behavior

The first step in the construction of the model is to define the proportion of the consumers in each of the six possible states as \( p_{ij} \) \( (i=1,2 \) and \( j=0,1,2) \). The subscript \( i \) represents the actual quality the consumer is buying, and \( j \) represents the quality the consumer thinks he is purchasing. The subscripts 1 and 2 indicate low and high quality, respectively, and the
subscript 0 indicates the state when the consumer has not yet made an
evaluation of the quality of the product he is buying. For example, $p_{12}$
represents the proportion of consumers who are buying the low quality
good but mistakenly identify it as a high quality good. Given these states,
the change in the proportion of consumers in each can be defined as a
function of the number of consumers already in various states; $s$, the birth
(new entrants to the market) and death (consumers who leave the market)
rate, which are assumed to be the same; $\lambda_1$, the amount of advertising by
the low and high quality firms, respectively; and $\eta$, the rate at which the
consumers "learn" of the product's quality. Learning occurs when the
consumer is able to make an assessment, based on his experience, of the
relative quality he has been purchasing. This learning involves an element
of inaccuracy. In this model, as in Rogerson [1981], the evaluation of the
product's quality is incorrectly made with probabilities $\alpha$ and $\beta$. These
mistake parameters are constrained so that

1. $\alpha + \beta < 1$.

This embodies the assumption that people are more likely to correctly
rather than incorrectly identify a product's quality.

Specifically, the change in the proportion of consumers in the states
is the sum of the net flow of consumers into and out of each of these
states, which the following set of equations describes:

2. $p_{10} = s/2 + \lambda_1 (p_{21} + p_{11}) - (\eta + s) p_{10}$
The rationale for these equations is as follows: because the new entrants have no prior knowledge of which firm offers the better quality product, the probability of a new entrant going to the high or the low quality firm is the same. Also since these consumers cannot yet have made an evaluation of the product they are buying, they can only enter the states where quality has not been learned. This is the source of the addition of the $s/2$ terms to equations 2 and 3. The deaths in each state are dependent on the number of consumers in that state, and hence the $-\delta p_{ij}$ term for each of the flow equations.

The rate of learning and the uncertainty of quality assessment enter the model in each of the equations. In the "unlearned" states, $\eta$ is a negative flow, representing the consumers who make an evaluation of the product's quality. In the "learned" states, the $\eta$ terms are positive flows from the unlearned states, and are multiplied by the probability of the consumer evaluating the quality correctly ($1 - \text{the mistake probability}$) or incorrectly (the mistake probability), depending on the state the consumer is entering.
The remaining terms in the equations result from when consumers believe they are buying low quality products, regardless of the accuracy of their belief. These consumers will want to leave the state they are in and reenter the market to find the higher quality product. The decision on which brand to purchase on reentry is determined by the advertising messages received by the consumers. These advertising messages may lead them to buy the same brand, by convincing them that they may have incorrectly assessed its quality, or it may lead them to buy the other brand because it has a higher quality. Consumers may receive either firm's advertising messages and will purchase that firm's product. This is considered a reentry into the state of not knowing what quality is being purchased, since the consumer evaluates quality only on the basis of his experience, and not on the basis of the advertisement. Therefore, the advertising to disgruntled customers are positive flows to the unlearned states and negative flows from the states where consumers believe the product is of low quality.

It should be noted that a consumer can only go from the unlearned state to a state of having learned, or exit the market. He cannot directly reenter the market in one of the unlearned states. Also, the consumer cannot go directly from a state of having learned that he is buying a low quality brand to a state of having learned he is buying a high quality brand. Rather, he must reenter the market in one of the unlearned states. Furthermore, once the consumer believes he is buying a high quality product, he will only leave that brand by exiting the market.

As in Mortensen [1985], equilibrium is defined to be when the flow
into and out of each state is the same, meaning that the size of each state is constant. This is when each of equations two through seven are equal to zero.

Firm Behavior

Firms are assumed to be profit maximizing. Their profit functions are specifically identified as:

\[ \pi_i = (\rho - c_i) p_i(\lambda_1, \lambda_2) - g(\lambda_i) \quad (i=1,2) \]

where \( \rho \) is the exogenously set price the firm receives per sale, \( c_i \) is the cost of producing a good of quality \( i \), and \( g(\lambda_i) \) is the cost of advertising amount \( l_i \), which is assumed to be linear and the same for both firms. The other term, \( p_i(\lambda_1, \lambda_2) \), is the number of customers at the firm. This number, derived from equations (2) through (7), is:

\[ p_1 = \frac{[1+\eta(1-\beta)/(s+\lambda_1+\lambda_2)+\eta\beta/s] \delta[(\eta+\delta)(s+\lambda_1+\lambda_2)+\eta\beta(\lambda_1-\lambda_2)]}{2(\eta+\delta)[s(s+\eta)+\lambda_1(s+\eta\beta)+\lambda_2(s+\eta(1-\alpha))]}
\]

\[ p_2 = \frac{[1+\eta\alpha/(s+\lambda_1+\lambda_2)+\eta(1-\alpha)/s] \delta[(\eta+\delta)(s+\lambda_1+\lambda_2)+\eta(1-\beta)(\lambda_2-\lambda_1)]}{2(\eta+\delta)[s(s+\eta)+\lambda_1(s+\eta\beta)+\lambda_2(s+\eta(1-\alpha))]}
\]

(For a complete derivation of these equations see Appendix 1.)
The $p - c$ term in the profit function is the profit per sale for the firm. It is assumed that

\[(11) \quad p - c_1 > p - c_2 > 0\]

which means the cost of producing the high quality good is more than the cost of producing the low quality good, and both are less than the price received from each sale. Multiplying this term by the number of customers the firm has gives the firm's revenue. Subtracting the cost of advertising results in the firm's profits.

Each firm selects an advertising strategy to maximize its profits. The firm will have to trade-off between the cost of advertising, which results in more customers, and not advertising, which saves money but brings in fewer customers.

Firms maximize profits with respect to their advertising levels when the first derivatives equal zero and the second derivatives are negative. However, given equations (9) and (10), determining the derivative of $p_i$ with respect to $\lambda_i$ is unmanageable. A reasonable simplification is to let $S$ go to zero. The rationale for this is that the rate at which people learn is much faster than the rate at which they are born and die. With this simplification, equations (9) and (10) become:

\[(9a) \quad p_1 = \frac{(1 + \alpha)\beta \lambda_1 + (1 - \alpha)\beta \lambda_2}{2[\lambda_1 \beta + \lambda_2(1 - \alpha)]} \]

\[(10a) \quad p_2 = \frac{(1 - \alpha)\beta \lambda_1 + (1 - \alpha)(2 - \beta)\lambda_2}{2[\lambda_1 \beta + \lambda_2(1 - \alpha)].}\]
Substituting this into the profit function, taking the partial derivatives with respect to $\lambda_i$, and setting them equal to zero (the first order conditions) results in:

\[
\begin{align*}
\partial \pi_1 / \partial \lambda_1 &= 0 = (\rho - c_1) \lambda_2 \beta (1 - \alpha) (1 + \alpha - \beta) / 2 [\lambda_1 \beta + \lambda_2 (1 - \alpha)]^2 - g'(\lambda_1) \\
\partial \pi_2 / \partial \lambda_2 &= 0 = (\rho - c_2) \lambda_1 \beta (1 - \alpha) (1 + \alpha - \beta) / 2 [\lambda_1 \beta + \lambda_2 (1 - \alpha)]^2 - g'(\lambda_2).
\end{align*}
\]

(For derivations of these equations, see Appendix 2.)

Solving these equations for $\lambda_1$ and $\lambda_2$ reveals the optimal advertising equilibria: one is the zero advertising level ($\lambda_1 = \lambda_2 = 0$ solves equations (12) and (13)) and the other is a positive advertising equilibrium described by:

\[
\begin{align*}
\lambda_1 &= (\rho - c_2) \beta (1 - \alpha) (1 + \beta + \alpha) / 2 g' [\beta + k(1 - \alpha)]^2 \\
\lambda_2 &= (\rho - c_1) \beta (1 - \alpha) (1 + \beta + \alpha) / 2 g' [\beta/k + (1 - \alpha)]^2 \\
\text{where } k &= (\rho - c_2) / (\rho - c_1).
\end{align*}
\]

(For derivations of these equations, see Appendix 3.)

**ANALYSIS**

**Advertising Levels**

The first result shown by this model is that, given the positive
advertising equilibrium, the low quality firm will advertise more than the high quality firm. Solving equations (12) and (13) for the ratio of advertising levels shows that

\[ \lambda_2 / \lambda_1 = (\rho - c_2) / (\rho - c_1). \]

(For the derivation of this equation see Appendix 3.)

From equation (11) we know that

\[ \frac{\rho - c_2}{\rho - c_1} < 1 \]

and consequently,

\[ \lambda_2 < \lambda_1. \]

This means that when there is a positive advertising equilibrium, the low quality firm will advertise more than the high quality firm. This result is particularly interesting because most of the literature has either assumed or tried to show that the high quality firms will advertise more than the low quality firms. One of the reasons the low quality firm will advertise more is that it has a higher probability of losing customers than the high quality firm, so it needs to advertise more heavily to attract customers. Also, though both firms want to advertise since it results in more customers, the high quality firm has a lower limit than the low quality firm on how much it can spend on advertising because the higher cost of producing the better product leaves less money per sale to spend on advertising.

Another result concerns the number of customers each of the firms has. In the no advertising equilibrium, substitution of \( \lambda_1 = \lambda_2 = 0 \) into equations (9) and (10) reveals that both firms will have the same number of customers. In the positive advertising equilibrium, the firm size is
dependent on the mistake parameters and the ratio of price minus cost parameters. The result is that

\[ p_2 > p_1 \text{ for } k > \alpha \beta / (1-\alpha)(1-\beta) \]

\[ p_1 > p_2 \text{ for } k < \alpha \beta / (1-\alpha)(1-\beta) \]

(For a derivation of these results, see Appendix 4.)

This means that when the ratio of the profit per high quality good sold to profit per low quality good sold is less than the ratio of the product of mistakes in learning to correct learning, the low quality firm will have more advertising. Obviously, higher mistake parameters would make this more likely. The better consumers can assess quality, it is more likely that the high quality firms will have more consumers.

Another aspect this model shows is which equilibrium the consumers and the firms prefer, and which is "stable". Because consumers pay the same price whether or not the firms advertise, and the advertisements provide them with information about the location and quality of the products they are buying, the buyers prefer positive advertising levels. The firms however, consider advertising to be a cost that reduces their profits. Because the number of customers is constant, advertising does not increase the sum of the two firm's profits. Therefore, the firms would be better off in the no advertising equilibrium.

However, each firm finds that it can increase its profits by advertising. Recall equations (1), (9a) and (10a). The number of customers each firm has is affected by both their own and the other firm's
advertising levels. However, each firm's proportion of customers increases with its own advertising. The numerator of \( p_1 \) increases with \( \lambda_1 \) at the rate of \( 1 + \alpha \) while the numerator \( p_2 \) increases with \( \lambda_1 \) at the rate \( 1 - \alpha \). Likewise, the numerator of \( p_2 \) increases with \( \lambda_2 \) at the rate of \( 2 - \beta \) while the numerator of \( p_1 \) increases with \( \lambda_2 \) at a rate of \( \beta \). This shows that the firm's share of the market is positively related to its own advertising and negatively related to the other firms advertising. Therefore, given the other firms advertising level, each firm believes that it can increase its profits by advertising more, up to the level where the cost of advertising absorbs the profits each firm makes by selling its product. In this way, the "stable" equilibrium is shown to be the positive advertising equilibrium.

CONCLUSIONS AND IMPLICATIONS

This paper has presented a model incorporating quality heterogeneity, price homogeneity, consumer learning with the possibility of mistakes, profit maximizing firms and advertising. The results of this model are that there can exist two advertising equilibria, one without advertising and one where both high and low quality firms advertise. The model shows why the positive advertising equilibrium exists most frequently, although the firms would maximize total profits by not advertising. The number of customers each firm has, and the amount of advertising by each firm, is dependent on the mistake parameters and the profit per sale of the
different qualities of the goods. In the positive advertising equilibrium the low quality firm always advertises more than the high quality firm.

One of the more interesting implications of this model is that the market results in positive advertising, which is beneficial to consumers. However, for the consumer to base his decision of which product to purchase on the amount of advertising by the firms would require him to be able to accurately determine which firm advertises the least, a very difficult proposition.
Appendix 1

Derivation of $p_1$, $p_2$

$$p_1 = p_{10} + p_{11} + p_{12}$$

$$p_2 = p_{20} + p_{21} + p_{22}$$

Solving equations (4)-(7) in terms of $p_{10}$ and $p_{20}$ and substituting gives

\[(a)\quad p_1 = \left(1 + \frac{\eta(1-\beta)}{\delta + \lambda_1 + \lambda_2} + \frac{\eta 0}{\delta} \right) p_{10}\]

\[(b)\quad p_2 = \left(1 + \frac{\eta \alpha}{\delta + \lambda_1 + \lambda_2} + \frac{\eta(1-\alpha)}{\delta} \right) p_{20}\]

Deriving $p_{10}$, by substituting in $p_{11}$ and $p_{21}$, gives

$$0 = \frac{\delta}{z} + \lambda_1 \left( \frac{\eta(1-\beta)p_{10} + \eta 0 p_{20}}{\delta + \lambda_1 + \lambda_2} \right) - (n+s)p_{10}$$

$$p_{10} = \frac{\delta(s + \lambda_1 + \lambda_2) + 2\lambda_1 \eta 0 p_{20}}{z \left[ (n+s)(s + \lambda_1 + \lambda_2) - \lambda_1 \eta(1-\beta) \right]}$$

Similarly

$$p_{20} = \frac{\delta(s + \lambda_1 + \lambda_2) + 2\lambda_2 \eta(1-\beta)p_{10}}{z \left[ (n+s)(s + \lambda_1 + \lambda_2) - \lambda_2 \eta 0 \right]}$$
substituting $P_{20}$ into $P_{10}$ and simplifying

$$P_{10} = \frac{s(s+\lambda; +\lambda_{z})[\eta s(s+\lambda; +\lambda_{z}) - \lambda_{z} \eta s] + \lambda; \eta s[s(s+\lambda; +\lambda_{z}) + z \lambda_{z} \eta s(1-\beta) P_{10}]}{z [\eta s(s+\lambda; +\lambda_{z}) - \lambda; \eta s(1-\beta)] [\eta s(s+\lambda; +\lambda_{z}) - \lambda_{z} \eta s]}$$

solving for $P_{10}$

$$P_{10} = \frac{s(s+\lambda; +\lambda_{z})[\eta s(s+\lambda; +\lambda_{z}) + \eta s(\lambda; -\lambda_{z})]}{z \eta s(s+\lambda; +\lambda_{z}) - \lambda; \eta s(1-\beta)} [\eta s(s+\lambda; +\lambda_{z}) - \lambda_{z} \eta s - \lambda; \eta s(1-\beta)] - z \lambda; \lambda_{z} \eta s^2 (1-\beta)$$

the denominator simplifies to

$$z \eta s(s+\lambda; +\lambda_{z}) [\eta s(s+\lambda; +\lambda_{z}) - \lambda_{z} \eta s - \lambda; \eta s(1-\beta)]$$

cancelling terms in the numerator and grouping $\lambda; s$ together,

$$P_{10} = \frac{s(s+\lambda; +\lambda_{z})[\eta s(s+\lambda; +\lambda_{z}) + \eta s(\lambda; -\lambda_{z})]}{z \eta s(s+\lambda; +\lambda_{z}) - \lambda; \eta s(1-\beta)} [\eta s(s+\lambda; +\lambda_{z}) - \lambda_{z} \eta s - \lambda; \eta s(1-\beta)] - z \lambda; \lambda_{z} \eta s^2 (1-\beta)$$

In the same manner, $P_{20}$ can be derived as:

$$P_{20} = \frac{s(s+\lambda; +\lambda_{z}) + \eta (1-\beta)(\lambda_{z} - \lambda;)}{z \eta s(s+\lambda; +\lambda_{z}) - \lambda; \eta s(1-\beta)} [\eta s(s+\lambda; +\lambda_{z}) + \eta (1-\beta)(\lambda_{z} - \lambda;)] - z \lambda; \lambda_{z} \eta s^2 (1-\beta)$$

Substituting these into (ia) and (ib) give equations (9) and (10).
Appendix 2

Derivation of $\frac{\partial \Pi_i}{\partial \lambda_i}$

From equation (8)

\[ \Pi_i = (\rho - c_i) p_i(\lambda_1, \lambda_2) - q(\lambda_1) \]
\[ \Pi_2 = (\rho - c_2) p_2(\lambda_1, \lambda_2) - q(\lambda_2) \]

The partial derivatives are

(iia) \[ \frac{\partial \Pi_i}{\partial \lambda_1} = (\rho - c_i) \frac{\partial p_i(\lambda_1, \lambda_2)}{\partial \lambda_1} - q'(\lambda_1) \]

(iib) \[ \frac{\partial \Pi_2}{\partial \lambda_2} = (\rho - c_2) \frac{\partial p_2(\lambda_1, \lambda_2)}{\partial \lambda_2} - q'(\lambda_2) \]

From equation (9a)

\[ p_i = \left( \frac{1}{2} \right) \frac{(1+\alpha)\beta \lambda_1 + (1-\alpha)\beta \lambda_2}{\lambda_1 \beta + \lambda_2 (1-\alpha)} \]

\[ \frac{\partial p_i}{\partial \lambda_1} = \left( \frac{1}{2} \right) \frac{(1+\alpha)\beta [\beta \lambda_1 + \lambda_2 (1-\alpha)] - \beta [(1+\alpha)\beta \lambda_1 + (1-\alpha)\beta \lambda_2]}{[\lambda_1 \beta + \lambda_2 (1-\alpha)]^2} \]

\[ = \left( \frac{1}{2} \right) \frac{\lambda_2 (1-\alpha)(1+\alpha)\beta - \beta^2 (1-\alpha)\lambda_2}{[\lambda_1 \beta + \lambda_2 (1-\alpha)]^2} \]

(iiia) \[ = \left( \frac{1}{2} \right) \frac{\lambda_2 (1-\alpha)\beta (1+\alpha-\beta)}{[\lambda_1 \beta + \lambda_2 (1-\alpha)]^2} \]
From equation (10a)

\[ P_2 = \left( \frac{1}{2} \right) \frac{(1-\alpha)\beta \lambda_1 + (1-\alpha)(2-\beta)\lambda_2}{\lambda_1 \beta + \lambda_2 (1-\alpha)} \]

\[ \frac{dP_2}{d\lambda_2} = \left( \frac{1}{2} \right) \frac{(1-\alpha) \left[ (2-\beta)[\lambda_1 \beta + \lambda_2 (1-\alpha)] - (1-\alpha)\beta \lambda_1 - (1-\alpha)(2-\beta)\lambda_2 \right]}{[\lambda_1 \beta + \lambda_2 (1-\alpha)]^2} \]

\[ = \left( \frac{1}{2} \right) \frac{(1-\alpha) \left[ (2-\beta)\beta \lambda_1 - (1-\alpha)\beta \lambda_1 \right]}{[\lambda_1 \beta + \lambda_2 (1-\alpha)]^2} \]

\[ (iiib) \quad = \frac{1}{2} \frac{\lambda_1 (1-\alpha)\beta (1+\alpha-\beta)}{[\lambda_1 \beta + \lambda_2 (1-\alpha)]^2} \]

Substituting (iia) and (iiib) into (iia) and (iib) results in equations (12) and (13)
Appendix 3

Solving for $\lambda_1, \lambda_2$ in equilibrium

Setting equations (12) and (13) equal to each other (both are equal to 0) gives

$$\left(\rho - c_1\right) \frac{\lambda_2/\beta(1-\alpha)(1-\beta+\alpha)}{2 \left[ \lambda_1/\beta + \lambda_2(1-\alpha) \right]^2} - g'(\lambda_1) = \left(\rho - c_2\right) \frac{\lambda_1/\beta(1-\alpha)(1-\beta+\alpha)}{2 \left[ \lambda_1/\beta + \lambda_2(1-\alpha) \right]^2} - g'(\lambda_2)$$

Because $g$ is defined as a linear function, $g'(\lambda_1) = g'(\lambda_2)$. Cancelling terms results in

$$\left(\rho - c_1\right) \lambda_2 = \left(\rho - c_2\right) \lambda_1$$

$$\frac{\lambda_2}{\lambda_1} = \frac{\rho - c_2}{\rho - c_1} = K$$

$$\lambda_2 = K \lambda_1$$

Substituting this into (12):

$$0 = \left(\rho - c_1\right) K \frac{\lambda_1/\beta(1-\alpha)(1-\beta+\alpha)}{2 \left[ \lambda_1/\beta + K \lambda_1(1-\alpha) \right]^2} - g'$$

$$g' = \left(\rho - c_1\right) K \frac{\beta(1-\alpha)(1-\beta+\alpha)}{2 \lambda_1 \left[ \beta + K(1-\alpha) \right]^2}$$

$$\lambda_1 = \left(\rho - c_1\right) K \frac{\lambda_1/\beta(1-\alpha)(1-\beta+\alpha)}{2 g' \left[ \beta + K(1-\alpha) \right]^2}$$
Substituting $\lambda_1 = \lambda_2 / k$ into (13)

$$0 = \frac{(\rho - c_2) \lambda_2 / k \beta (1 - \alpha)(1 - \beta + \alpha)}{Z [\beta \lambda_2 / k + \lambda_2 (1 - \alpha)]^2} - g'$$

$$g' = \frac{(\rho - c_2) \beta (1 - \alpha)(1 - \beta + \alpha)}{2k \lambda_2 [\beta / k + \lambda_2 (1 - \alpha)]^2}$$

$$\lambda_2 = \frac{(\rho - c_2) \beta (1 - \alpha)(1 - \beta + \alpha)}{2k g' [\beta / k + \lambda_2 (1 - \alpha)]^2}$$

Substituting $k = \frac{\rho - c_2}{\rho - c_1}$ results in equations (14) and (15)
Appendix 4

Customers per firm in no advertising equilibrium

Further simplification of (9) and (10):

\[ P_1 = \left[ \frac{S(S+\lambda_1+\lambda_2) + \eta (1-\beta)S + \eta \beta (S+\lambda_1+\lambda_2)}{S+\lambda_1+\lambda_2} \right] \]

\[ P_2 = \left[ \frac{S(S+\lambda_1+\lambda_2) + \eta \alpha S + n (1-\alpha)(S+\lambda_1+\lambda_2)}{S+\lambda_1+\lambda_2} \right] \]

Setting \( \lambda_1 = \lambda_2 = 0 \):

\[ P_1 = \left[ \frac{S^2 + \eta (1-\beta)S + \eta \beta S}{S} \right] \left[ \frac{(n+\delta)S}{2(n+\delta)S(S+n)} \right] \]
\[ P_1 = \left[ \delta + n(1-\beta) + \eta/\beta \right] \frac{1}{2(n+s)} \]

\[ = \frac{1}{2} \]

\[ P_2 = \left[ \frac{S^2 + \eta \alpha S + n(1-\alpha)S}{S} \right] \frac{(n+s)S}{2(n+s)S(s+n)} \]

\[ = \left[ \frac{S + \eta \alpha + n(1-\alpha)}{S} \right] \frac{1}{2(n+s)} \]

\[ = \frac{1}{2} \]

\[ P_1 = P_2 \]
Customers per firm in positive advertising equilibrium.

Substituting $\lambda_2 = k \lambda_1$ into (9a) and (10a):

(9a) \[ P_1 = \frac{(1+\alpha) \beta \lambda_1 + (1-\alpha) \beta k \lambda_1}{2[\lambda_1 \beta + \lambda_2 (1-\alpha)]} \]

(10a) \[ P_2 = \frac{(1-\alpha) \beta \lambda_1 + (1-\alpha) (2-\beta) k \lambda_1}{2[\lambda_1 \beta + \lambda_2 (1-\alpha)]} \]

\[ P_1 \succ P_2 \]

\[ (1+\alpha) \beta \lambda_1 + (1-\alpha) \beta k \lambda_1 \succ (1-\alpha) \beta \lambda_1 + (1-\alpha) (2-\beta) k \lambda_1 \]

\[ (1+\alpha) \beta + (1-\alpha) \beta k \succ (1-\alpha) \beta + (1-\alpha) (2-\beta) k \]

\[ \frac{(1+\alpha) - (1-\alpha) \beta}{2 \alpha \beta} \succ \frac{(1-\alpha) (2-\beta) - (1-\alpha) \beta}{2 \alpha \beta} \]

\[ \frac{2 \alpha \beta}{1-\alpha} \succ \frac{2 \beta \alpha}{1-\alpha} \succ \frac{\beta \alpha}{(1-\alpha) (1-\beta)} \succ k \]
Selected Bibliography


